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AIAA 81-4246

Sensitivity of Modal-Space Control to Nonideal Conditions

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Introduction

THE theoretical concept of active modal-space control, with application proposed for flexible space structures, has been developed recently by Meirovitch and Oz.[1-4] The present authors have investigated modal-space control from the perspective of a similar technology, modal vibration testing of structures, and have conducted numerical studies in an attempt to assess the practical applicability of the method.[5,6]

Much of the previous study of modal-space control has been based on certain ideal conditions: knowledge of the exact modal parameters of the structure to be controlled; availability of an essentially unlimited number of control actuators; and the capability of measuring without error the response contribution of an individual mode of vibration. Since these ideal conditions generally do not exist in reality, the practical value of modal-space control depends on its effectiveness under more realistic, nonideal conditions.

Physical and Modal Equations of Motion

References 1-4 employ structure discretization by means of the global assumed modes method. However, we assume here discretization by the more direct finite-element method, for which the basic time-dependent coordinates are displacements of selected grid points on the structure.

The linear deformational and rigid body dynamics of an undamped structure with no gyroscopic members can be described by the physical equations of motion

$$m\ddot{q}+kq=c \tag{1}$$

Square matrices m and k are mass and stiffness matrices, respectively; $q(t)$ is the vector of all grid-point displacements (translations and rotations); and $c(t)$ is the vector of all grid-point control actions (forces and moments). Disturbance excitation is omitted because it has no role in this analysis.

The simplify the analysis, we neglect internal passive structural damping. For the slight structural damping which might be expected of large space structures, this should lead in most cases to conservative predictions of overall (i.e., active plus passive) damping.

We denote the dimension of the vectors in Eq. (1) as N and specify simply that N be as large as necessary to accurately model the dynamics of the structure. Since m and k are assumed to be developed by finite-element analysis, the maximum computationally practical dimension is on the order of a few thousand.

Let us assume that n normal modes, accounting for both rigid body and deformational motion, result from eigenanalysis of Eq. (1) with null right-hand side, that these calculated modes accurately represent the modes of the real structure, and that the number n of modes is adequate to describe all disturbed motion of practical importance. (The possible consequences of inaccurate modal information are discussed in a subsequent section.) We denote the natural frequency of the jth mode as ω_j and the $N\times n$ modal matrix as Φ, the jth column of which ϕ_j is the physical mode shape of the jth mode. The orthogonality conditions are $\Phi^T m\Phi = M = \text{diag}\ (M_1, M_2, \ldots, M_n)$ and $\Phi^T k\Phi = K = \text{diag}\ (K_1, K_2, \ldots, K_n)$, where $K_j = M_j\omega_j^2$. The standard modal transformation of variables is $q=\Phi\xi$, where $\xi(t)$ is the vector of n modal coordinates. With this transformation, premultiplication by Φ^T, and the orthogonality conditions, Eq. (1) becomes

$$M\ddot{\xi}+K\xi=\Phi^T c \tag{2}$$

Modal-Space Control with a Limited Number of Actuators

There can be only a finite number n_a of control actuators, which for a real structure is probably smaller than n. We consider the actuators to be applied at points and in directions on the structure corresponding to a specific subset q^a of degrees of freedom. Accordingly, the $n_a \times 1$ actuator submatrix of c is denoted as c^a, all other elements of c being zero. Hence, in Eq. (2), $\Phi^T c = \Phi^{aT} c^a$, where the $n_a \times n$ matrix Φ^a consists of the appropriate rows of Φ.

Control vector c^a depends on the measured motion, so it can be considered in general a function of all ξ_j's and/or their derivatives. To simplify the analysis, we consider only velocity feedback, so that $c^a = c^a(\dot{\xi})$.

The essence of modal-space control is selection of c^a so as to decouple as much as possible of the right-hand side of Eq. (2). To do this, we first designate n_c specific modes as the modes to be controlled, the remaining n_r modes being the uncontrolled or residual modes. In practice, we would expect $n_a \le n_c < n$. The subset of Eq. (2) describing the controlled modes is

$$M^c\ddot{\xi}^c + K^c\xi^c = \Phi^{acT} c^a(\dot{\xi}) \tag{3}$$

where superscript c denotes appropriate partitions of the matrices in Eq. (2). In particular, the $n_a \times n_c$ matrix Φ^{ac} consists of the columns of Φ^a associated with the controlled modes.

Now we seek for a controlled mode, say mode s, a time-independent shape, c_s^{ac}, of the control vector which will isolate that mode from all other controlled modes. ("Force apportioning" is a term which has been used to describe this process in multiple-shaker modal vibration testing.) Next, we form the total control vector as a linear sum of vectors for all individual controlled modes,

Received Jan. 23, 1981.

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$$c^a(\dot{\xi}) = \sum_s^{n_c} c_s^{ac} f_s^c(\dot{\xi}_s) = C^{ac} f^c \tag{4}$$

where C^{ac} is an unknown (at this point) $n_a \times n_c$ matrix. The use of function $f_s^c(\dot{\xi}_s)$ in this form presumes the capability of measuring without error the individual modal response $\dot{\xi}_s$. An idealized physical measurement scheme giving this condition is explained in the next section, where it is shown that, for control having the character of modal viscous damping,[1,3,4] we can write $f^c = \Phi_i^c \dot{\xi}^c$, Φ_i^c being a nonsingular diagonal matrix to be defined. With Eq. (4), Eq. (3) now becomes

$$M^c \ddot{\xi}^c + K^c \xi^c = \Phi^{acT} C^{ac} \Phi_i^c \dot{\xi}^c \tag{5}$$

In order for Eq. (5) to be fully uncoupled and to have a specified degree of artificial viscous damping in each controlled mode, it is necessary that

$$\Phi^{acT} C^{ac} \Phi_i^c = -D^c \tag{6}$$

where $D^c = \text{diag}\,(2M_s \zeta_s \omega_s,\ s \text{ over controlled modes})$, and ζ_s is the viscous damping factor specified for mode s.

If $n_c > n_a$, then Eq. (6) for C^{ac} can be satisfied only approximately by means of the matrix pseudoinverse.[6] But if $n_c = n_a$, then the exact solution is

$$C^{ac} = -\Phi^{ac-T} D^c \Phi_i^{c-1} \tag{7}$$

provided that actuators and controlled modes are selected so that Φ^{ac} is not singular.

If the number of actuators equals the number of controlled modes and if Eq. (7) is satisfied, then the controlled modes are fully uncoupled and independently controlled. But the residual modes are excited by modal-space control if $n_c < n$, for the full set of modal equations, Eq. (2), becomes

$$\begin{bmatrix} M^c & 0 \\ 0 & M^r \end{bmatrix} \begin{bmatrix} \ddot{\xi}^c \\ \ddot{\xi}^r \end{bmatrix} + \begin{bmatrix} K^c & 0 \\ 0 & K^r \end{bmatrix} \begin{bmatrix} \xi^c \\ \xi^r \end{bmatrix} = \begin{bmatrix} -D^c \\ \Phi^{arT} C^{ac} \Phi_i^c \end{bmatrix} \dot{\xi}^c \tag{8}$$

where superscript r denotes matrix partitions associated with the residual modes. Equation (8) leads to bounded response of the residual modes, a characteristic of modal-space control that Meirovitch and Oz[1] have described. Analysis and numerical simulations[5,6] show that the bounded response can be small or, on the other hand, so great as to be quite unacceptable. The level of residual response for a particular structure is dependent on the specific actuators and controlled modes selected.

Consider, for example, the case in which at time $t=0$ deformational potential energy E_{mo} exists in only one of the controlled modes, say mode m. From Eq. (8), motion in mode m is suppressed as $t \to \infty$, and all other controlled modes remain undisturbed; but motion in each residual mode becomes a nonzero, steady-state sinusoidal oscillation at that mode's natural frequency. A transient analysis[5,6] shows that $E_{p\infty}/E_{mo} \propto (\phi_p^{aT} c_m^{ac})^2$, where $E_{p\infty}$ is the steady-state energy remaining in mode p. Since the distribution of control suitable for suppressing mode m is not, in general, orthogonal to the shape of mode p, residual energy is almost inevitable.

Consequences of Imperfect Modal Response Measurements

The following is a conceptually simple approach for measuring individual responses of controlled modes. A velocity sensor is placed at a degree of freedom, say the ith, which has nonzero response for every controlled mode. The output of this sensor is

$$\dot{q}_i(t) = \sum_{j=1}^{n} \dot{q}_{ij}$$

where, from the modal transformation, $\dot{q}_{ij} = \Phi_{ij} \dot{\xi}_j$. The sensor output is directed into parallel narrow-band filters, one for each controlled mode centered at that mode's natural frequency. The ideal filter for a mode would introduce no phase lag and would have a sufficiently narrow passband and sufficiently steep rolloffs to completely exclude response from all other modes.

If sensing is exact and filtering is ideal, then the output of the filter for mode s is $f_s^c = \dot{q}_{is}$. The same form of processing for each controlled mode in conjunction with control actuation apportioning, Eq. (7), wherein $\Phi_i^c = \text{diag}\,(\Phi_{is},\ s$ over controlled modes), leads to complete isolation of each controlled mode, Eq. (8).

Now consider the same feedback control process, but with relaxation of the ideal filtering condition so that each narrowband filter is considered to have realistic values for bandwidth and rolloff rates. The output of the filter for mode s may now include contributions from all modes; we express it as

$$f_s^c = \sum_{j=1}^{n} A_{sj} \dot{q}_{ij}$$

where A_{sj} is the attenuation of a signal at frequency ω_j by the filter centered on ω_s. In this case, then, $f^c = A\Phi_i \dot{\xi}$, where A is the $n_c \times n$ matrix of attenuation constants and $\Phi_i = \text{diag}\,(\Phi_{i1}, \Phi_{i2}, \ldots, \Phi_{in})$. It is appropriate to partition A into $n_c \times n_c$ and $n_c \times n_r$ submatrices, $A = [A^c \,|\, A^r]$, and also to define diagonal matrix Φ_i^r, where $\Phi_i = \text{block diag}\,(\Phi_i^c, \Phi_i^r)$. Then with Eq. (4), Eq. (3) for the controlled modes becomes

$$M^c \ddot{\xi}^c + K^c \xi^c = \Phi^{acT} C^{ac} (A^c \Phi_i^c \dot{\xi}^c + A^r \Phi_i^r \dot{\xi}^r) \tag{9}$$

If $n_c = n_a$, then the controlled modes can be decoupled from each other by control actuation apportioning

$$C^{ac} = -\Phi^{ac-T} D^c \Phi_i^{c-1} A^{c-1} \tag{10}$$

provided that the inverse matrices exist. But observation spillover coupling from the residual modes inevitably results from a nonnull A^r. Therefore, it is to be expected that nonideal filtering will reduce the effectiveness of modal-space control and might even produce system instability. Reference 6 presents the results of a limited numerical study of the effects of nonideal filtering. For the models studied, $n=36$, $n_c = 6$ (including three rigid body modes), and $n_a = 6$. Actuation apportioning Eq. (7), not Eq. (10), was used. One of the examples considered is a structure having a pair of very closely spaced natural frequencies among the controlled modes, a condition which one might expect to be especially sensitive to nonideal filtering. The numerical results do not confirm this pessimistic expectation, though, for all cases with filter characteristics typical of actual hardware exhibit neither substantive degradation of damping performance, relative to corresponding cases with ideal filtering, nor serious instability.

The structure of A is of interest. With modes numbered in order of increasing natural frequencies, the filter for mode s

gives

$$0 \leq \ldots \leq A_{s,s-1} \leq A_{s,s} \geq A_{s,s+1} \geq \ldots \geq 0 \tag{11}$$

where $A_{s,s} = 1$ for a unity gain filter properly centered on ω_s.[6] If the controlled modes are the n_c lowest modes, and if high-performance filters are used, then Eq. (11) suggests that A^c will, in effect, be banded and A^r will, in effect, be primarily null except for a populated region in its lower left-hand corner.

We note that Meirovitch and Oz[1,3] have proposed an alternative approach to modal response measurement which requires, in general, several motion sensors but no filtering.

Consequences of Inaccurate Modal Information

The decoupling effectiveness of control actuator apportionings, Eqs. (7) and (10), depends on the accuracy of modal matrix Φ^{ac}. If the true modal matrix differs from the matrix used to calculate C^{ac}, which will generally be the case in practice, then decoupling will be incomplete. Moreover, the effectiveness of Eq. (10) depends additionally on the accuracy of the natural frequencies used for computation of A.

Reference 6 includes a limited numerical study of the reduction in effectiveness of Eq. (7) due to inaccurate modal information. The modal matrix of a "model" structure was used in the calculation of actuator apportionings to be applied on a similar "actual" structure. The differences between the "model" and "actual" structures were designed to be representative of differences that often exist between a finite-element modal and the actual hardware being analyzed. For the cases studied, the imperfect actuator apportionings produce only slight reductions in control effectiveness relative to corresponding cases with perfect apportionings. The results suggest that it would take substantial errors in mode shape estimates to render modal-space control ineffective or to produce serious instability.

Concluding Remarks

The most important factor limiting the effectiveness of modal-space control is control spillover into the residual modes which occurs because there are fewer control actuators than modes. However, if residual mode excitation can be acceptably minimized by a judicious choice of controlled modes and actuators, then the effectiveness of modal-space control appears to be reasonably insensitive to inaccurate modal parameters and to observation spillover resulting from realistic signal filtering.

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AIAA 81-4247

Time-Varying Weights for Optimal Control with Inequality Constraints

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Nomenclature

a_p	= applied acceleration, ft/s^2
a_t	= target acceleration, ft/s^2
b	= weight given to energy term in performance index
b_0	= initial value of b
E	= total mean-squared control energy, g^2-s^2
J	= performance index with constant weight
J'	= performance index with time-varying weight
R_1	= component of measurement error variance, rad^2-s
R_2	= contribution to measurement error variance from glint, rad^2-s^3
t	= time
t_f	= intercept time
v	= cross-range velocity, ft/s
V_x	= along-range velocity, ft/s
w	= process noise, ft/s^3
y	= cross-range displacement, ft
ϵ	= measurement noise, rad-s$^{1/2}$
σ	= observable angle, rad
τ	= target maneuver time constant, s

Introduction

INEQUALITY constraints, such as engine thrust limits or maximum permissible deflections of aerodynamic control surfaces, are encountered quite often in control system design. When these constraints prohibit linear quadratic Gaussian (LQG) formulations with Riccati equation solutions, the following characteristics of linear optimization solutions are sacrificed: 1) determination of optimal time-varying Kalman filter and closed-loop feedback gains, uniquely expressible in terms of the same type of formulation used for the optimal (Kalman) estimator; and 2) straightforward characterization for statistical performance of both estimator and controller.

In many applications, especially those involving stochastic processes, motivation for recovering these benefits is understandably great. A common approach is to omit the inequality constraint from the formulation while including, in the performance index to be minimized, a time integral of the constrained variable weighted by a constant. The resulting linear optimization is then repeatedly performed with different weightings, until that variable (or, for stochastic problems, a chosen multiple of its rms value) instantaneously reaches its maximum or minimum permissible level at some point in the solution. The performance index in that case is optimized without violating the constraint, but the time integral term in the optimization criterion does not provide a potentially beneficial lingering of the constrained variable near its extremal value, along an extremal arc, in the vicinity of the point just described. This prolonged hovering near an extremal value, generally associated with saturation nonlinearity, is achieved here using a linear formulation via time-varying weights.

Received Sept. 9, 1980; revision received Feb. 27, 1981.

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